5.4 Factor Quadratic Expressions of the Form $x^2 + bx + c$

A water garden combines a pond with aquatic plants and often ornamental fish, such as koi, to add visual appeal to the landscape. The area of a rectangular water garden can be represented by the **quadratic expression** $x^2 + 5x + 6$. To find the length and the width of the rectangle, you can write the trinomial as the product of two binomials.

quadratic expression

- a second-degree polynomial
- $4x^2 + 20$ and $x^2 + 7x + 10$ are quadratic expressions

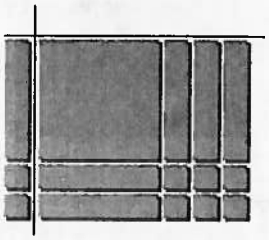
Tools

- algebra tiles

Investigate A

How can you use a model to factor quadratic expressions of the form $x^2 + bx + c$?

1. To factor $x^2 + 5x + 6$, use algebra tiles to create a rectangular area whose length and width represent the factors of the trinomial.
   a) Arrange one x^2-tile, five x-tiles, and six unit tiles to form a rectangle with area $x^2 + 5x + 6$. Place tiles along the left side and top to find the length and width of the rectangle. One dimension has been done for you.



   b) Write the equation for the trinomial as a product of the binomial dimensions.

2. Repeat step 1 for each trinomial.
   a) $x^2 + 6x + 5$ b) $x^2 + 3x + 2$
   c) $x^2 + 4x + 3$ d) $x^2 + 6x + 8$

3. Each trinomial in steps 1 and 2 is of the form $x^2 + bx + c$. What do you notice about b and c and the binomial factors for each trinomial? Describe the relationship.

4. Test your conclusions from step 3 on each trinomial. Use algebra tiles to check your answer.
   a) $x^2 + 7x + 6$ b) $x^2 + 8x + 12$

5. Reflect Describe a process for finding the factors of a quadratic expression of the form $x^2 + bx + c$.

vestigate B

w can you use patterns to factor quadratic expressions of the m $x^2 + bx + c$?

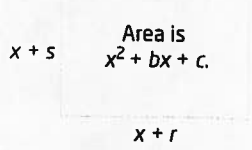
A. Positive Values of b and c

1. Expand and simplify each product. Try to apply the distributive property mentally.
   - **a)** $(x + 4)(x + 3)$
   - **b)** $(x + 1)(x + 5)$
   - **c)** $(x + 7)(x + 8)$
2. The result of expanding each binomial product of the form $(x + r)(x + s)$ in step 1 is a trinomial of the form $x^2 + bx + c$. Describe how you calculated b and c using the values of r and s.
3. Use the patterns from step 2 to reverse the process. Write each trinomial of the form $x^2 + bx + c$ as a binomial product of the form $(x + r)(x + s)$.
   - **a)** $x^2 + 6x + 8$
   - **b)** $x^2 + 7x + 10$
   - **c)** $x^2 + 9x + 20$
   - **d)** $x^2 + 10x + 21$
4. Describe a process for factoring quadratic expressions of the form $x^2 + bx + c$.

B: Negative Values of b and/or c

1. Expand and simplify each product. Try to apply the distributive property mentally.
   - **a)** $(x - 3)(x - 2)$
   - **b)** $(x - 1)(x - 5)$
   - **c)** $(x - 1)(x + 5)$
   - **d)** $(x + 3)(x - 8)$
2. The result of expanding each binomial product of the form $(x + r)(x + s)$ in step 1 is a trinomial of the form $x^2 + bx + c$.
   - **a)** Describe how you determined the signs of the values of b and c when both values of r and s were negative.
   - **b)** Describe how you determined the signs of the values of b and c when only one of the values of r and s was negative.
3. Use your process to factor each trinomial.
   - **a)** $x^2 - 7x + 10$
   - **b)** $x^2 + 4x - 5$
   - **c)** $x^2 - 4x - 5$
   - **d)** $x^2 - 3x - 10$
4. How does your process for factoring quadratic expressions of the form $x^2 + bx + c$ change when the values of b and/or c are negative?

By finding the dimensions of a rectangle whose area is a quadratic expression, you are reversing the process of expanding two binomials that you learned in Section 5.1. This process is called factoring.



Another way to factor a quadratic expression of the form $x^2 + bx + c$ is to study the patterns from multiplying two binomials.

$$\begin{aligned}(x + r)(x + s) &= x^2 + sx + rx + rs \\ &= x^2 + rx + sx + rs \\ &= x^2 + (r + s)x + rs\end{aligned}$$

Therefore, $x^2 + bx + c = (x + r)(x + s)$, where $r + s = b$ and $r \times s = c$.

In general, you will factor *over the integers*, meaning that the values of r and s are integers only.

Many quadratic expressions, such as $x^2 + 3x + 5$, cannot be factored over the integers. No two integers have a product of 5 and a sum of 3.

Example 1 Factor Quadratic Expressions

Factor, if possible.

a) $x^2 + 7x + 12$

b) $x^2 + 4x + 6$

c) $x^2 - 29x + 28$

d) $x^2 + 3x - 18$

e) $x^2 - 4x - 21$

Solution

a) For $x^2 + 7x + 12$, $b = 7$ and $c = 12$. Use a table to find two integers whose product is 12 and whose sum is 7. In order to have a positive product and a positive sum, both numbers must be positive.

Factors of 12	Product	Sum
1, 12	12	13
2, 6	12	8
3, 4	12	7

Therefore, r is 3 and s is 4.
$x^2 + 7x + 12 = (x + 3)(x + 4)$

b) For $x^2 + 4x + 6$, $b = 4$ and $c = 6$.

Factors of 6	Product	Sum
1, 6	6	7
2, 3	6	5

I need to find two positive integers whose product is 6 and whose sum is 4.

Since no two integers have a product of 6 and sum of 4, $x^2 + 4x + 6$ cannot be factored over the integers.

c) For $x^2 - 29x + 28$, $b = -29$ and $c = 28$.

Factors of 28	Product	Sum
[illegible]	[illegible]	[illegible]
−2, −14	28	−16
−4, −7	28	−11

I need to find two integers whose product is 28 and whose sum is −29. To have a positive product and a negative sum, both numbers must be negative.

Therefore, r is -1 and s is -28.
$x^2 - 29x + 28 = (x - 1)(x - 28)$

d) For $x^2 + 3x - 18$, $b = 3$ and $c = -18$.

Factors of −18	Product	Sum
1, −18	−18	−17
−1, 18	−18	17
2, −9	−18	−7
−2, 9	−18	7
3, −6	−18	−3
[illegible]	[illegible]	[illegible]

I need to find two integers whose product is −18 and whose sum is 3. To have a negative product and a positive sum, one number must be negative and the other positive.

Therefore, r is -3 and s is 6.
$x^2 + 3x - 18 = (x + 6)(x - 3)$

e) For $x^2 - 4x - 21$, $b = -4$ and $c = -21$.

Factors of −21	Product	Sum
1, −21	−21	−20
−1, 21	−21	20
[illegible]	[illegible]	[illegible]
−3, 7	−21	4

I need to find two integers whose product is −21 and whose sum is −4. To have a negative product and a negative sum, one number must be negative and the other positive.

Therefore, r is 3 and s is -7.
$x^2 - 4x - 21 = (x + 3)(x - 7)$

Example 2 Dimensions of a Water Garden

a) Determine binomials that represent the dimensions of the rectangular water garden.

b) Determine the dimensions if x represents 1 m.

Area is $x^2 + 5x + 6$.

Solution

a) Factor the quadratic expression for the area. Find two integers whose product is 6 and whose sum is 5. The integers are 2 and 3.
$x^2 + 5x + 6 = (x + 2)(x + 3)$
The dimensions can be represented by $x + 2$ and $x + 3$.